Journal of Statistical Physics, Vol. 55, Nos. 1/2, 1989

Existence of the Transfer Matrix Formalism for a Class of Classical Continuous Gases

Roman Gielerak[1]

Received August 3, 1987; revision received October 26, 1988

For classical gases of particles interacting through nonnegative, many-body interactions of short range it is verified that the corresponding grand canonical Gibbs measures have the global Markov property for sufficiently low values of the chemical activity. This yields the existence of a (nonsymmetric in general) transfer matrix formalism for such systems.

KEY WORDS: Classical gas; grand canonical Gibbs ensemble; Gibbs measure; Kirkwood–Salsburg operator; global Markov property; transfer matrix.

1. INTRODUCTION

One of the fundamental tools used to study the classical statistical mechanical systems that live on lattices in R^d is the transfer matrix formalism. It is an expecially useful tool wherever the interactions in the given lattice systems are short-ranged. The fruitful applications of this method include solving exactly the thermodynamics for a large class of one- and two-dimensional lattice spin systems.[(1)] Modest application yields the input to study the particle spectrum properties in the lattice gauge theories.[(2)] There are many other important applications of the transfer matrix formalism. From this follows the importance of establishing the possibility of using transfer matrix analysis to extract some information about the system under consideration.

Compared to the lattice case there are many fewer results on the transfer matrix formalism for a continuous system of classical statistical

[1] Research Center Bielefeld-Bochum-Stochastics, University of Bielefeld, D-4800 Bielefeld 1, Federal Republic of Germany, and Institute of Theoretical Physics, University of Wroclaw, 50-205, Wroclaw, Poland.

0022-4715/89/0400-0183$06.00/0

mechanics (including the Euclidean field theory models). The existence of the symmetric transfer matrix for the Euclidean field theory can be deduced from Osterwalder–Schrader positivity.[3] However, with regard to the theory of classical gases, there is no such kind of positivity (excluding some special cases). There are some results establisting the existence of the transfer matrix formalism for the classical gas theory, but they describe either finite-volume situations,[4] one-dimensional gas,[5,6] or neutral systems of particles interacting via some special two-body, sufficiently regular potential of positive type.[7]

In this paper I prove the existence of the transfer matrix formalism for a large class of classical gases in which the interaction is given by an arbitrary, repulsive many-body potential of short range. The proof I present here is very simple. I obtain the proof by verifying that the grand canonical ensemble Gibbs measure has the global Markov property. This yields the existence of the Markov process with values in the space of locally finite configurations that are located in a certain slice of the space R^d. The transition function of this process plays the role of the transfer matrix and the grand canonical Gibbs equilibrium measure plays the role of the path space measure of the corresponding process. The restriction to the pure repulsive interactions is the price we have to pay for a very simple verification of the global Markov property presented here. Such a property of the Gibbs measure is not easy to verify already on the level of classical spin systems.[32] For the verification of the global Markov property in the case of lattic systems see refs. 9–13 and 32, and for continous systems see refs. 14–17.

The method used in this paper to verify the global Markov property of the corresponding Gibbs measure is a simple analysis of the corresponding Kirkwood–Salsburg-like operators using some elementary facts from the general theory of the dual pair of Banach spaces.

To extend the result to the arbitrary many-body stable interactions one has to work with an adaptation of the Dobrushin theory[18,19] to this case and it is much harder than the method presented here. Such an extension will be presented elsewhere.

This paper is organized as follows. In Section 2, I collect some basic definitions from the theory of classical gases and formulate the result in a precise way. Section 3 includes the details of the proof for the case of two-body interactions. This is done for a pedagogical exposition the our method. Sections 4 and 5 contain extensions of the method of the proof given in Section 3 to treat the general many-body interactions. In the Appendix I present some technical points necessary to complete the proof of main result of this paper, stated as Theorem 1 in Section 2.

2. PRELIMINARY DEFINITIONS AND FORMULATION OF THE RESULT

2.1. Classical Gases[21, 22]

Let Ω be the collection of all finite or countable subsets of R^d having no limit points in R^d. Ω is provided with the weakest topology τ in which the map

$$\pi_\Omega: \qquad \Omega \ni \omega \to \omega(\Lambda) \equiv \omega \cap \Lambda \in \Omega_f(\Lambda) \tag{2.1}$$

in continuous for any open, bounded, Borel subset $\Lambda \subset R^d$, where $\Omega_f(R^d)$ [respectively $\Omega_f(\Lambda)$] is the collection of all finite subsets $\omega \subset R^d$ (resp. $\omega \subset \Lambda$) with the point-to-point convergence topology τ_f. The σ-algebra(s) corresponding to τ_f ($\equiv$ Borel algebra) will be denoted by $\mathscr{F}_f(\Omega_f)$ [resp. $\mathscr{F}_f(\Omega_f(\Lambda)) \equiv \mathscr{F}_f(\Lambda)$].

The pair (Ω, τ) then forms a polish space.[23] The corresponding Borel σ-algebra is denoted by $\mathscr{F}(R^d)$. In a similar fashion one defines the σ-algebras $\mathscr{F}(\Lambda)$, where Λ is an arbitrary, Borel subset of R^d. Clearly, $\mathscr{F}(\Lambda_1) \subset \mathscr{F}(\Lambda_2)$ provided $\Lambda_1 \subset \Lambda_2$ and moreover for $\Lambda_1 \cup \Lambda_2 = \Lambda$ with $\Lambda_1 \cap \Lambda_2 = \varnothing$ one has $\mathscr{F}(\Lambda_1) \simeq \mathscr{F}(\Lambda_1) \otimes \mathscr{F}(\Lambda_2)$.

Let us define on the σ-algebra $\mathscr{F}_f(R^d)$ the measure

$$\hat{\pi}_0^z(A) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \operatorname{vol}_n\{(x_1,\ldots, x_n) \in R^n / \{x_1,\ldots, x_n\} \in A\} \tag{2.2}$$

$$\hat{\pi}_0^z(\phi) = 1 \qquad \text{and} \qquad z \geqslant 0$$

This is the noninteractinig-gas Poisson measure with the chemical activity z.

Now, let $V = (V_1, V_2,\ldots, V_k,\ldots)$ be a sequence of functions, each V_k being defined as a symmetric, measurable function on R^{dk}. With the help of V we then define

$$E_V: \quad \Omega_f \times \Omega_f \to R_1 \cup \{+\infty\}$$

by

$$E_V(\omega_2 \mid \omega_2) = \begin{cases} \displaystyle\sum_{\substack{\omega \subset \omega_1 \vee \omega_2, \\ \omega \cap \omega_1 \neq \varnothing, \\ \omega \cap \omega_2 \neq \varnothing}} V_{|\omega|}(\omega) & \text{if } \omega_1 \neq \varnothing, \quad \omega_2 \neq \varnothing \\ 0 & \text{otherwise} \end{cases} \tag{2.3}$$

where $|\omega| = \operatorname{card} \omega$. For a given $\omega \in \Omega$, denote by $\omega(\Lambda)$ its restriction to the set $\Lambda \subset R^d$. For $\eta \in \Omega_f$ and $\omega \in \Omega$ let us define

$$E_V(\eta \mid \omega) = \lim_{n \to \infty} E_V(\eta \mid \omega(\Lambda_n)) \tag{2.4}$$

where $(\Lambda_n)_n$ is an arbitrary, monotonic sequence of bounded subsets of R^d such that $\bigcup_n \Lambda_n = R^d$, if such limit exists. Now let $\Lambda \subset R^d$ be bounded and let $\omega \in \Omega$; denote

$$Z_\Lambda^{\omega(\Lambda^c)}(z, \beta) = \int_{\Omega_f(\Lambda)} \hat{\pi}_0^z(d\eta) \exp - E_V(\eta \mid \eta \vee \omega(\Lambda^c)) \tag{2.5}$$

$$\mu_\Lambda^{\omega(\Lambda^c)}(z, \beta \mid A) = [Z_\Lambda^{\omega(\Lambda c)}(z, \beta)]^{-1} \times \int_{\pi_\Omega{}^{-1}(A)} \hat{\pi}_0^z(d\eta) \exp -E_V(\eta \mid \eta \vee \omega(\Lambda^c)) \tag{2.6}$$

for any $A \in \mathscr{F}(\Lambda)$, if the limits and integrals under consideration exist.

Any probabilistic measure $\mu(z, \beta)$ on $\{\Omega, \mathscr{F}(R^d)\}$ is called the grand canonical Gibbs measure corresponding to the interaction V, chemical activity z, and (inverse) temperature $\beta > 0$, iff:

gcG (i) The limit (2.4) exists for almost every (w.r.t. the measure $\pi_0^z \otimes \mu$) pair $(\eta, \omega) \in \Omega_f(R^d) \otimes \Omega(R^d)$.

gcG (ii) In the sense of measures we have

$$\mu \circ \mu_\Lambda^{(-)}(z, \beta \mid \cdot) = \mu(z, \beta)(\cdot) \tag{2.7}$$

for every bounded $\Lambda \subset R^d$, where $(-)$ is the integration variable. The set of such measures is denoted by $\mathscr{G}(z, \beta, V)$. It is known that for the case of suitably regular interactions the set $\mathscr{G}(z, \beta, V)$ is nonempty for any $z \geqslant 0$ and $\beta \geqslant 0$.[21,24] With some additional restrictions on V some uniqueness theorems are known.[25,26] Finally, in the case of superstable interactions it is known that it is possible to select some special subset of $\mathscr{G}(z, \beta, V)$ of the so-called tempered Gibbs measures which has the structure of a Choquet simplex. This was originally pointed out by Ruelle[21] for the case of two-body interactions and then extended to arbitrary many-body superstable interactions in ref. 28.

2.2. The Results

In this paper it is assumed that $V_k \geqslant 0$ for every k and moreover each V_k is short-ranged, i.e., $\exists d_k < \infty$:

$$\forall_{(x)_k = (x_1, \dots, x_k)} : \quad V_k((x)_k) = 0 \quad \text{unless} \quad \max_{ij}\{\text{dist}(x_i, x_j)\} < d_k \tag{2.8}$$

Moreover, it is assumed that $d^* = \sup_k d_k < \infty$. Then one has the following result.

Theorem 0. Assume that V is a nonnegative many-body interaction of shortrange, say $d^* < \infty$. Then, for any $z < \exp[-2V_d(d^*)]$, where $V_d(d^*)$ is the volume of the d-dimensional ball of radius d^*, the set $\mathscr{G}(z, \beta, V)$ consists of exactly one element $\nu_\infty(z, \beta \mid -)$.

The proof of this theorem is a byproduct of the proof of the following Theorem 1 and therefore will be not explained here. However, I would like to point out the simplicity of the arguments used, especially in the uniqueness part of it. If follows from the very definition of the set $\mathscr{G}(z, \beta, V)$ that every element of it fulfills the so-called local Markov property, which means that for every bounded domain $\Lambda \subset R^d$ we have

$$E_\mu\{F \cdot G \mid \mathscr{F}(\partial_{d^*}(\Lambda))\} = E_\mu\{F \mid \mathscr{F}(\partial_{d^*}(\Lambda))\} \cdot E_\mu\{G \mid \mathscr{F}(\partial_{d^*}(\Lambda))\} \quad (2.9)$$

where $\mu \in \mathscr{G}(z, \beta, V)$, F and G are bounded random variables measurable with respect to the σ-algebra $\mathscr{F}(\Lambda)$, respectively, $\mathscr{F}(\Lambda^c - \partial_{d^*}(\Lambda))$, where $\partial_{d^*}(\Lambda) = \{x \notin \Lambda \mid \operatorname{dist}(x, \Lambda) \leqslant d^*\}$, and $E_\mu\{- \mid -\}$ denotes conditional expectation values. The main result of this paper consists in showing that the unique Gibbs measure $\nu_\infty(z, \beta \mid -)$ from Theorem 0 has the Markov property (2.9) also for unbounded domains $\Lambda \subset R^d$. This is the so-called the global Markov property.

As is well known, uniqueness or extremality of the locally Markov Gibbs measure does not yield automatically the global Markov property. For the discussion of this phenomenum see refs. 29–31. As was pointed out by Föllmer,[32] in order to verify the global Markov property we have to show the so-called strong uniqueness for the measure $\nu_\infty(z, \beta)$, which means the following. Let $\Lambda \subset R^d$ be an arbitrary domain in R^d such that $R^d - \partial\Lambda = \Omega_+ \cup \Omega_-$, where Ω_+ and Ω_- are the only two connected components. Let $\partial^+_{d^*}(\Lambda) = \{x \notin \Omega_- \mid \operatorname{dis}(x, \Omega_-)_- < d^*\}$ and let $(\Lambda_n)_n$ be any sequence of bounded subsets of R^d which tends to R^d monotonously and by inclusion. Then the strong uniqueness means that we have the equality

$$\begin{aligned} &\lim_{n \to \infty} E_{\nu_\infty(z,\beta)}\{- \mid \mathscr{F}(\partial^+_{d^*}(\Lambda)) \vee \mathscr{F}(\Lambda^c_n)\} \\ &\quad = E_{\nu_\infty(z,\beta)}\{- \mid \mathscr{F}(\partial^+_{d^*}(\Lambda))\} \qquad \nu_\infty(z, \beta)\text{-almost surely} \qquad (2.10) \end{aligned}$$

This is the strategy of the proof. Exactly such a strategy has been used in previous proofs of the global Markov property for a different class of systems in refs. 7, 10, 14–17, and 32.

Theorem 1. Assume that V is a nonnegative many-body interaction of range $d^* < \infty$. Let $z < \exp[-2V_d(d^*)]$. Then the unique grand canonical Gibbs measure $\nu_\infty(z, \beta)$ has the global Markov property.

Let $\Sigma_0 = \{x = (x^0, \underline{x}) \in R^d \mid x^0 = 0\}$ and let $\partial_{d^*}(\Sigma_0) = \{x \in R^d \mid 0 \leqslant x^0 \leqslant d^*\}$. Then we have the following corollary.

Corollary. The exists a stationary Markov process ξ_t with the state space $\Omega(\partial_{d_*}(\Sigma_0))$ such that $\nu_\infty(z, \beta)$ is its path space measure. The Markov semigroup k_t associated to ξ_t has a unique eigenvector which corresponds to the eigenvalue with the largest real part and this eigenvalue is separated by the nonzero gap from the rest of the spectrum.

Remark. For $d=1$ it can be extracted from ref. 33 that Theorem 1 is valid for every $z \geqslant 0$.

To fix the idea, I restrict the proofs to the particular hyperplane Σ_0. But it is evident that the proofs work also for any hypersurface $\Sigma \subset R^d$, thus yielding the proof of the global Markov property for $r_\infty(z, B)$.

3. PROOF OF THEOREM 1. THE CASE OF TWO-BODY INTERACTIONS

In this section I specialize to the case when $V=(0, V_2, 0, 0,...)$, i.e., to the case of two-body interaction. I assume that V_2 is spherically symmetric (for simplicity) as a function on R^d with compact support of size $r<\infty$, i.e., $V_2(|x|)=0$ if $|x|>r$. In the following I drop the subscript 2. This restriction is of pedagogical interest only since one has to deal then with a well-known Kirkwood–Salsburg operator for two-body interactions. In Section 5 I generalize the analysis below to the general multibody situation.

The finite volume Λ, conditioned by ω at $\partial_r(\Sigma_0)$ and by $\omega'(\Lambda^c)$ at Λ^c, Gibbs measure $\nu_\Lambda(-|\mathscr{F}(\partial_r(\Sigma_0)) \vee \mathscr{F}(\Lambda^c)(\omega \vee \omega')$, is described completely by its correlation functions

$$\{\rho_\Lambda^{\omega(\partial_r(\Sigma_0)\cap\Lambda)\vee\omega'(\Lambda^c)}(z, \beta\,|\,(\dot{x})_n)\}_{n=1,...,\infty}$$

which are given by the following formulas:

$$\begin{aligned}
&\rho_\Lambda^{\omega(\partial_r(\Sigma_0)\cap\Lambda)\vee\omega'(\Lambda^c)}(z, \beta\,|\,(x)_n)\\
&\quad=(Z_\Lambda^{\omega(\Lambda\cap\partial_r(\Sigma_0))\vee\omega'(\Lambda^c)}(z, \beta))^{-1}\\
&\qquad\times\chi_{\Lambda-\partial_r(\Sigma_0)}(x)_n\sum_{m=0}^{\infty}\frac{z^{m+n}}{m!}\int_{\Lambda-\partial_r(\Sigma_0)}d(\underline{y})_m\\
&\qquad\times\exp-\beta\mathscr{E}((x)_n\vee(\underline{y})_m\,|\,(x)_n\vee(\underline{y})_m\vee\omega(\partial_r(\Sigma_0)\cap\Lambda)\vee\omega'(\Lambda^c))
\end{aligned}\tag{3.1}$$

$$\begin{aligned}
&Z_\Lambda^{\omega(\Lambda\cap\partial_r(\Sigma_0))\vee\omega'(\Lambda^c)}(z, \beta)\\
&\quad=\int_{\Omega(\Lambda-\partial_r(\Sigma_0))}\pi_0^z(d\eta)\exp-\beta\mathscr{E}(\eta(\Lambda)\,|\,\omega(\partial_r(\Sigma_0)\cap\Lambda)\vee\omega'(\Lambda^c))
\end{aligned}\tag{3.2}$$

where π_0^z is the normalized version of $\hat{\pi}_0^z$; for $(x)_n \cap \Lambda \cap \partial_r(\Sigma_0) \neq \varnothing$ and for $(x)_n \cap \Lambda \cap \omega(\partial_r(\Sigma_0)) \neq \varnothing$ they are defined by

$$\rho_\Lambda^{\omega(\partial_r(\Sigma_0) \cap \Lambda) \vee \omega'(\Lambda^c)}(z, \beta \mid (x)_n)$$
$$= \begin{cases} 0 & \text{if} \quad (x)_n \cap \Lambda \cap \omega(\partial_r(\Sigma_0)) \neq (x)_n \\ 1 & \text{if} \quad (x)_n \cap \Lambda \cap \omega(\partial_r(\Sigma_0)) = (x)_n \end{cases} \tag{3.3}$$

The measure $\nu_\infty(z, \beta \mid \mathscr{F}(\partial_r(\Sigma_0))(\omega)$ conditioned at the σ-algebra $\mathscr{F}(\partial_r(\Sigma_0))$ is uniquely determined by the thermodynamic limit (any of) of the correlation functions $\{\rho_\Lambda^{\omega(\partial_r(\Sigma_0))}(z, \beta \mid (x)_n)\}$ with $\omega'(\Lambda^c) = \varnothing$.

To verify the global Markov property of the measure $\nu_\infty(z, \beta)$ it is sufficient to prove that all the thermodynamic limits of $\{\rho_\Lambda^{\omega(\partial_r(\Sigma_0) \cap \Lambda) \vee \omega'(\Lambda^c)}(z, \beta \mid (x)_n)\}$ as ω' varies over the set Ω are the same as that with $\omega' = \varnothing$.

Let B_ξ be the Banach space of sequences of measurable functions $f = (f_n)_{n=1,2,\ldots}$, where each f_n: $R^{nd} \to R^1$, equipped with the norm

$$\|f\|_\xi = \sup_n \xi^{-n} \operatorname*{esssup}_{(x)_n \in R^{nd}} |f_n(x)_n| \tag{3.4}$$

Throughout this section I choose $\xi = 1$ (but see Section 4). To simplify the notation I will sometimes use the following abbreviations: $\Gamma_\Lambda(\omega) \equiv \omega(\partial_r(\Sigma_0) \cap \Lambda)$, $\Gamma_\infty(\omega) \equiv \Gamma_{\Lambda = R^d}(\omega)$, $\Gamma_\Lambda \equiv \Lambda - \partial_r(\Sigma_0)$, and $\Gamma_\infty = R^d - \partial_r(\Sigma_0)$.

Let

$$\mathbb{K}^{\Gamma_\infty(\omega)}: \quad \mathscr{B}_1 \to \mathscr{B}_1 \tag{3.5}$$

be defined by

$$(\mathbb{K}^{\Gamma_\infty(\omega)} f)_m (x)_m$$
$$\equiv \exp -\beta \mathscr{E}(x_1 \mid (x)'_m \vee \Gamma_\infty(\omega)) \left\{ f_{m-1}((x)'_{m-1}) \right.$$
$$\left. + \sum_{n=1}^\infty \frac{1}{n!} \int_{R^d - \partial_r(\Sigma_0)} d(y)_n \, K(x_1 \mid (y)_n f_{m+n-1}((x)'_m \vee (y)_n) \right\} \tag{3.6}$$

for $m > 1$; and for $m = 1$

$$(\mathbb{K}^{\Gamma_\infty(\omega)} f)_1 (x)_1 = \exp -\beta \mathscr{E}(x_1 \mid \Gamma_\infty(\omega))$$
$$\times \sum_{m=1}^\infty \frac{1}{m!} \int_{R^d \setminus \partial_r(\Sigma_0)} d(y) \, K(x_1 \mid (y)_m) f_m((y)_m) \tag{3.7}$$

Here

$$(x)'_m = (x_2, \dots, x_m)$$
$$K(x \mid y) = e^{-\beta V(x-y)} - 1 \tag{3.8}$$
$$K(x \mid (y)_m) = \prod_{i=1}^{m} K(x \mid y_i)$$

For a set $\Lambda \subset R^d$ let

$$\Pi_\Lambda: \ \mathscr{B}_1 \to \mathscr{B}_1$$
$$(\Pi_\Lambda f)_n(x)_n = \prod_{i=1}^{n} \chi_\Lambda(x_i) f_n((x)_n) \tag{3.9}$$

Then the correlation functions of the conditioned measure $v_\infty(z, \beta \mid \mathscr{F}(\partial_d(\Sigma_0)))(\omega)$ fulfills the following identities:

$$\Pi_{\Gamma_\infty} \rho_\infty^{\Gamma_\infty(\omega)}(z, \beta) = z \mathbb{K}^{\Gamma_\infty(\omega)} \Pi_{\Gamma_\infty} \rho_\infty^{\Gamma_\infty(\omega)}(z, \beta) + z \Pi_{\Gamma_\infty} \alpha_\infty(\Gamma_\infty(\omega)) \tag{3.10}$$

where

$$\rho_\infty^{\Gamma_\infty(\omega)}(z, \beta) = \{\rho_\infty^{\Gamma_\infty(\omega)}(z, \beta \mid ((x)_n)\}_{n=1,2} \tag{3.11}$$

$$\alpha_\infty(\Gamma_\infty(\omega)) = (\exp -\beta \mathscr{E}(x_1 \mid \Gamma_\infty(\omega)), 0, \dots, 0, \dots) \tag{3.12}$$

I check that the identities (3.10) in a unique way determine $\rho_\infty^{\Gamma_\infty(\omega)}(z, \beta)$ at least for small z and moreover $\rho_\infty^{\Gamma_\infty(\omega)}(z, \beta)$ are limits of the corresponding finite-volume quantities $\rho_\Lambda^{\Gamma_\infty(\omega)}(z, \beta)$ where the limit is understood componentwise and locally uniform (i.e., uniform on compact sets). This is proved by the standard contraction map principle applied to the identities (3.10) and its corresponding finite-volume versions.

Lemma 3.1. Let $V_d(r)$ be the volume of a d-dimensional ball with radius r.

1. For $|z| < \exp[-2V_d(r)]$ the operator $\mathbb{K}^{\Gamma_\infty(\omega)} \Pi_{\Gamma_\infty(\omega)}$ is contractive in the space $\mathscr{B}_1$, uniformly in $\omega \in \Omega$, and the relation (3.10) uniquely determines the conditioned correlation functions $\rho_\infty^{\Gamma_\infty(\omega)}(z, \beta)$.

2. For $|z| < \exp[-2V_d(r)]$ the finite-volume conditioned correlation functions $\rho_\Lambda^{\Gamma_\infty(\omega)}(z, \beta)$ tend componentwise and locally uniformly to the corresponding infinite-volume quantities $\rho_\infty^{\Gamma_\infty(\omega)}$. This holds for every $\omega \in \Omega$.

The proof will be omitted as not differing from that presented in ref. 34, Theorem 4.2.3.

The space $\mathscr{B}_1$ is the weak dual space to the Banach space $*\mathscr{B}_1$ consisting of sequences of measurable functions $(\varphi)=(\varphi_n)_{n=1,2,\dots}$ equipped with the norm

$$*\|\varphi\|_1=\sum_{n=1}^{\infty}\int d(\underline{x})_n\,|\varphi_n((x)_n)| \tag{3.13}$$

Then the pair of spaces $(*\mathscr{B}_1, \mathscr{B}_1)$ forms a dual pair of Banach spaces.[35] Now using a certain argument from the theory of dual pairs, I will prove the following result, which is key for the verification of the global Markov property of $\mu_\infty(z,\beta)$.

Proposition 3.2. Let $|z|<\exp[-2V_d(r)]$. Then for any $\omega',\omega\in\Omega$ fixed, we have

$$\lim_{\Lambda\uparrow R^d}\rho_\Lambda^{\Gamma_\Lambda(\omega)\vee\omega'(\Lambda^c)}(z,\beta)=\rho_\infty^{\Gamma_\infty(\omega)}(z,\beta) \tag{3.14}$$

where the limit in (3.4) is taken componentwise and is uniform on compact subsets.

Proof. The correlation functions $\rho_\Lambda^{\Gamma_\Lambda(\omega)\vee\omega'(\Lambda^c)}$ fulfill the following identity:

$$\rho_\Lambda^{\Gamma_\Lambda(\omega)\vee\omega'(\Lambda^c)}(z,\beta)=z\mathbb{K}_\Lambda^{\Gamma_\Lambda(\omega)\vee\omega'(\Lambda^c)}\Pi_{\Gamma_\Lambda}\rho_\Lambda^{\Gamma_\Lambda(\omega)\vee\omega'(\Lambda^c)}$$
$$+z\alpha_\Lambda(\Gamma_\Lambda(\omega)\vee\omega(\Lambda^c)) \tag{3.15}$$

where the operator $\mathbb{K}_\Lambda^{\Gamma_\Lambda(\omega)\vee\omega'(\Lambda^c)}$ is defined by the formula

$$(\mathbb{K}_\Lambda^{\Gamma_\Lambda(\omega)\vee\omega'(\Lambda^c)}f)_m(x)_m$$
$$=\exp-\beta\mathscr{E}(x_1\,|\,(x)'_m\vee\Gamma_\Lambda(\omega)\vee\omega'(\Lambda^c))\left\{f_{m-1}((x)'_m)\right.$$
$$\left.+\sum_{n=1}^{\infty}\frac{1}{n!}\int_{(\Lambda-\partial_d(\Sigma_0))^{\otimes n}}d(y)_n\,K(x_1\,|\,(y)_n)\,f_{n+m-1}((x)'_m\vee(y)_n)\right\} \tag{3.16}$$

and similarly for $m=1$

$$(\mathbb{K}_\Lambda^{\Gamma_\Lambda(\omega)\vee\omega'(\Lambda^c)}f)_1(x_1)$$
$$=\exp-\beta\mathscr{E}(x_1\,|\,\Gamma_\Lambda(\omega)\vee\omega'(\Lambda^c))$$
$$\times\left\{1+\sum_{m=1}^{\infty}\int_{(\Lambda-\partial_d(\Sigma_0))^{\otimes m}}d(y)_m\,K(x_1\,|\,(y)_m)\,f_m(y)_m\right\} \tag{3.17}$$

Operators $K_\Lambda^{\Gamma_\Lambda(\omega) \vee \omega'(\Lambda^c)}$ and $K^{\Gamma_\infty(\omega)}$ have the forms

$$\mathbb{K}_\Lambda^{\Gamma_\Lambda(\omega) \vee \omega'(\Lambda^c)} = \exp -\beta\mathscr{E}(x_1 \mid \Gamma_\Lambda \vee \omega'(\Lambda^c)) \cdot \mathbb{K} \cdot \Pi_{\Gamma_\Lambda} \tag{3.18}$$

where the operator $\exp[-\beta\mathscr{E}(\cdot \mid \cdot)]$ acts as multiplication and the generic operator $\mathbb{K}$ acts in $\mathscr{B}_1$ according to (3.17) and (3.18).

The dual $*\mathbb{K}$ of the operator $\mathbb{K}$ in the dual pair $(\mathscr{B}_1, *\mathscr{B}_1)$ can be easily calculated, with the result

$$(*\mathbb{K}\psi)_m(x_m) = \sum_{k=0}^{m} \frac{1}{k!} \int_{R^d} dy\, K(y \mid (x)_k)\, \Psi_{1+m-k}(y \vee (x)_m - (x)_k) \tag{3.19}$$

Now we observe that

$$\begin{aligned} &*\| *(\mathbb{K}^{\Gamma_\infty(\omega)} - \mathbb{K}_\Lambda^{\Gamma_\Lambda(\omega) \vee \omega'(\Lambda^c)})\psi \|_1 \\ &\quad \leqslant *\| *\mathbb{K}(\exp -\beta\mathscr{E}(x_1 \mid (x) \vee \Gamma_\infty(\omega))[\exp -\beta\mathscr{E}(x_1 \mid \omega'(\Lambda^c))](\Pi_{\Gamma_\Lambda} - 1)\psi \|_1 \\ &\quad \leqslant \|\mathbb{K}^{\Gamma_\infty(\omega)}\|_1 \, *\| \{\exp[-\beta\mathscr{E}(x_1 \mid \omega'(\Lambda^c))] - 1\} \, \Pi_{\Gamma_\Lambda}\psi \|_1 \end{aligned} \tag{3.20}$$

Now it is not difficult to note that for any $\omega' \in \Omega^*$, any compact $\Delta \subset R^d - \partial_r(\Sigma_0)$, and any short-ranged potential V, we have

$$\lim_{\Lambda \uparrow R^d} \|\mathscr{E}(x \mid \omega'(\Lambda^c))\|_{L^\infty(\Delta)} = 0 \tag{3.21}$$

whenever $\Lambda \uparrow R^d$ in an appropriate way. From this and the last line (3.20) it follows easily that

$$\forall_{\omega' \in \Omega} : \quad s\text{-}\lim_{\Lambda \uparrow R^d} *\mathbb{K}_\Lambda^{\Gamma_\Lambda(\omega) \vee \omega'(\Lambda^c)} = *\mathbb{K}^{\Gamma_\infty(\omega)} \tag{3.22}$$

The strong convergence of (3.22) yields the strong convergence of the resolvents:

$$s\text{-}\lim_{\Lambda \uparrow R^d} (1 - z\, *\mathbb{K}_\Lambda^{\Gamma_\Lambda(\omega) \vee \omega'(\Lambda^c)})^{-1} = (1 - z\, *K_\infty^{\Gamma_\infty(\omega)})^{-1} \tag{3.23}$$

assuming that the corresponding resolvents exists, which is true for $|z| < \exp[-2V_d(r)]$.

Now we are ready to prove

$$\forall_{\omega' \in \Omega} : \quad *\text{-}\omega \lim_{\Lambda \uparrow R^d} \rho_\Lambda^{\Gamma_\Lambda(\omega) \vee \omega'(\Lambda^c)}(z, \beta) = \rho_\infty^{\Gamma_\infty(\omega)}(z, \beta) \tag{3.24}$$

assuming $|z| < \exp[-2V_d(r)]$. For this, note

$$\begin{aligned}
\forall_{\psi \in {}^*\mathscr{B}_1} : \quad & \langle \psi, \rho_\Lambda^{\Gamma_\Lambda(\omega) \vee \omega'(\Lambda^c)}(z, \beta) - \rho_\infty^{\Gamma_\infty(\omega)}(z, \beta) \rangle \\
&= z\langle \psi, (1 - z\mathbb{K}_\Lambda^{\Gamma_\Lambda(\omega) \vee \omega'(\Lambda^c)})^{-1} \alpha_\Lambda(\Gamma_\Lambda(\omega) \vee \omega'(\Lambda^c)) \\
&\quad - (1 - z\mathbb{K}^{\Gamma_\infty(\omega)})^{-1} \alpha(\Gamma_\infty(\omega)) \rangle \\
&= z\langle \psi, [(1 - z\mathbb{K}_\Lambda^{\Gamma_\Lambda(\omega) + \omega'(\Lambda^c)})^{-1} - (1 - z\mathbb{K}_\Lambda^{\Gamma_\infty(\omega)})^{-1} \alpha_\infty(\omega(\Gamma_\infty(\omega))] \rangle \\
&\quad + z\langle \psi, (1 - z\mathbb{K}_\Lambda^{\Gamma_\Lambda(\omega) \vee \omega'(\Lambda^c)})^{-1} [\alpha_\Lambda(\Gamma_\Lambda(\omega) \vee \omega'(\Lambda^c)) - \alpha_\infty(\Gamma_\infty(\omega))] \rangle \\
&= z\langle [(1 - z\, {}^*\mathbb{K}^{\Gamma_\Lambda(\omega) \vee \omega'(\Lambda^c)})^{-1} - (1 - z\, {}^*\mathbb{K}^{\Gamma_\infty(\omega)})^{-1}] \psi, \alpha_\infty(\Gamma_\infty(\omega)) \rangle \\
&\quad + z\langle (1 - z\, {}^*K_\Lambda^{\Gamma_\Lambda(\omega) \vee \omega'(\Lambda^c)})^{-1} \psi, \alpha_\Lambda(\Gamma_\Lambda(\omega) \vee \omega'(\Lambda^c)) - \alpha_\infty(\Gamma_\infty(\omega)) \rangle
\end{aligned} \tag{3.25}$$

which shows the claimed convergence if we additionally note

$$\lim_{\Lambda \uparrow R^d} \alpha_\Lambda(\Gamma_\Lambda(\omega)) \vee \omega'(\Lambda^c) = \alpha_\infty(\Gamma_\infty(\omega)) \tag{3.26}$$

in the sense of L_∞.

I have used the identity

$$^*(1 - zA)^{-1} = (1 - z^*A)^{-1} \tag{3.27}$$

which holds in any dual pair of Banach spaces (see, e.g., ref. 35, §22, §1.7).

Now we have to sharpen the proved *-weak convergence to the locally uniform, componentwise one. But this can be done by using the Mayer–Montroll identities. The details of this (presumable well-known) procedure are explained in the Appendix. QED

4. STABLE INTERACTION WITH HARD-CORE CUTOFF

In this section I will try to relax the assumption that V_2 is pure repulsive. However, in this case several complications arise, the main one of which is the question about bounds uniform in the boundary data on the corresponding Kirkwood–Salsburg operators.

For stable interactions we have to choose a certain modification of the Kirkwood–Salsburg operator $\mathbb{K}^{\Gamma_\infty(\omega)}$, as was pointed out by Ruelle.[34] Let $\mathscr{M}$ be an index juggling operator choosing the x_1 coordinate in such a way that for every $n \geq 1$ we have

$$\mathscr{E}((x)_1 \,|\, (x)'_n) \geq -2B \tag{4.1}$$

where B is the stability constant of the potential V. Then the modified Kirkwood–Salsburg operator $\tilde{\mathbb{K}}^{\Gamma_\infty(\omega)}$ is defined as the superposition

$$\tilde{\mathbb{K}}^{\Gamma_\infty(\omega)} = \mathbb{K}^{\Gamma_\infty(\omega)} \circ \mathscr{M} \tag{4.2}$$

The first question arising is whether the operator $\mathscr{M}$ has a bounded dual in the pair $({}^*\mathscr{B}_\xi, \mathscr{B}_\xi)$.

Lemma 4.1. There exists a bounded linear operator ${}^*\mathscr{M}$ in the space ${}^*\mathscr{B}_\xi$ such that

$${}^*\mathscr{M} = (\mathscr{M})^*$$

Moreover, the operator ${}^*\mathscr{M}$ is given by the formula

$${}^*\mathscr{M} = \hat{v}_1 \sum_{j=1}^{\infty} \pi_j \tag{4.3}$$

where the operator $\hat{v}_1$ and π_j are defined in the following way. Let

$$\Omega_i(m) := \{(x)_m \in R^d - \partial_d(\Sigma_0) \mid \mathscr{E}((x_i) \mid (x)_m - (x_{ik})) \geqslant -2B\} \tag{4.4}$$

and let $\chi_i(m)$ be the corresponding characteristic function of the set $\Omega_i(m)$. Then we define

$$\hat{v}_1(m) = \chi_i(m) \left[\sum_{i=1}^{m} \chi_i(m) \right]^{-1} . \tag{4.5}$$

The operators π_j are defined as multiplication operators by $\Theta(m-j+1)$, where the Θ function is defined by $\Theta(k)=0$ for $k \leqslant 0$ and $\Theta(k)=1$ for $k>0$.

Proof. By a simple calculation it is easy to check that

$$\forall_{\substack{f \in \mathscr{B}_\xi \\ \varphi \in {}^*\mathscr{B}_\xi}} : \langle \varphi, \mathscr{M} f \rangle = \langle {}^*\mathscr{M} \varphi, f \rangle \quad \text{QED}$$

Remark. For similar calculations see ref. 36. I remark that the existence of the bounded dual operator in the dual pair does not follow automatically and needs to be proved.

As a conclusion we have that the dual operator ${}^*\tilde{\mathbb{K}}^{\omega(\partial_{d^*}(\Sigma_0))}$ to the modified KS operator $\tilde{\mathbb{K}}^{\omega(\partial_{d^*}(\Sigma_0))}$ exists. However, in order to apply the method of Section 3 to control the corresponding limits, one has to find estimates which are uniform in the boundary datas. More precisely, at present one does not known whether the KS identity

$$\rho_\infty^{\omega(\partial_{d^*}(\Sigma_0))}(z, \beta) = \tilde{\mathbb{K}}^{\omega(\partial_{d^*}(\Sigma_0))} \rho_\infty^{\omega(\partial_{d^*}(\Sigma_0))} + z\alpha_\infty(\omega(\partial_{d^*}(\Sigma_0)) \tag{4.6}$$

determines the corresponding conditioned correlation functions in a unique way. The result is formulated as follows.

Proposition 4.2. For every $\omega, \omega' \in \Omega$ there exists a number $C(\omega)$ such that whenever $z < C(\omega)$, we have

$$*\text{-}z \lim_{\Lambda \uparrow R^d} \rho_\Lambda^{\omega(\partial_{d^*}(\Sigma_0)) \vee \omega'(\Lambda^c)}(z, \beta) = \rho_\infty^{\omega(\partial_{d^*}(\Sigma_0))}(z, \beta) \tag{4.7}$$

But it is particularly easy to choice such a sequence of $\omega_n \in \Omega$ that our estimates gives $\lim_{n \to \infty} C(\omega_n) = 0$.

In order to overcome this problem, let us introduce the cutoff of the possible density of particles in $\partial_{d^*}(\Sigma_0)$. This is obtained by adding to the potential V_2 a hard-core potential V^{HC} defined by

$$V_r^{\text{HC}}(x) = \begin{cases} +\infty & \text{for } |x| \leqslant r \\ 0 & \text{otherwise} \end{cases} \tag{4.8}$$

Certainly we assume that $r < d^* = \text{diam}(\text{supp } V_2)$. Then the admissible particle configurations are those for which the minimal distance between particles is greater then r. It follows that we should consider the whole problem on the restricted configuration space Ω^r defined as a Borel subset of Ω and composed of those configurations which fulfill the requirement stated above.

The application of the methods of Section 3, Lemma 3.1 then leads to the proof of the following theorem.

Theorem 4.1. For the potential of the form $V = V_2 + V_r^{\text{HC}}$ where V_2 is a short-ranged, finite, regular, stable potential with range equal to d^* and stability constant B, and for

$$|z| \leqslant C_r(\beta)^{-1} e^{O(r, d^*, B)}$$

where

$$C_r(\beta) = 2V_d(r) \int_r^{d^*} |e^{-\beta V(x)} - 1| \, dx$$

$$O(r, d^*, B) = -2\beta\, B - \left(\frac{{}^1V_d(d^*)}{2V_d(r)}\right) \beta \sup |V_2(x)|/-1, \qquad x \in [r, d^*]$$

there exists a unique grand canonical Gibbs measure $\nu_\infty(z, \beta)$ on the space Ω_r. This unique Gibbs measure possesses the global Markov property.

5. GENERAL, MANY-BODY REPULSIVE INTERACTIONS

In this section I extend the proof to the general case of an interaction given by a sequence of many-body potentials $(V_k)_{k=1,2,...}$ where each V_k is defined on Ω_k and is almost surely nonnegative there. Moreover, letting d_k^* denote the smallest number with the property

$$\underset{\substack{\omega \in \Omega_k \\ \omega = (x_1, ..., x_k)}}{\forall} : \quad V_k(\omega) = 0 \quad \text{unless} \quad \max_{i,j}\{\text{dist}(x_i, x_j)\} < d_k^*$$

let us assume that

$$d^* = \sup_k d_k^* < \infty \tag{5.1}$$

The corresponding, finite-volume Λ, grand canonical Gibbs measure $v_\Lambda(d\eta \mid \mathscr{F}(\partial_d(\Sigma_0)) \vee \mathscr{F}(\Lambda^c)\}(\omega, \omega')$ contitioned by $\omega \in \Omega(\partial_{d^*}(\Sigma_0))$ at $\mathscr{F}(\partial_{d^*}(\Sigma_0))$ and by $\omega' \in \Omega(\Lambda^c)$ at the σ-algebra $\mathscr{F}(\Lambda^c)$ is described completely by its correlation functions, which are defined by the formula

$$\begin{aligned} &\rho_\Lambda^{\Gamma_\Lambda(\omega) \vee \omega'(\Lambda^c)}(z, \beta \mid (x)_n) \\ &\quad = [Z_\Lambda^{\Gamma_\Lambda(\omega) \vee \omega'(\Lambda^c)}(z, \beta)]^{-1} \\ &\quad \times \sum_{k=0}^{\infty} \frac{z^{k+m}}{k!} \int_{\Lambda - \partial_d(\Sigma_0)} d(y)_k \, \chi_{\Lambda - \partial_d(\Sigma_0)}(x)_n \\ &\quad \times \exp[-\beta\mathscr{E}((y)_k \vee (x)_n \mid (y)_k \vee (x)_n \vee \Gamma_\Lambda(\omega) \vee \omega'(\Lambda^c))] \end{aligned} \tag{5.2}$$

where now

$$\mathscr{E}(\omega \mid \omega') = \sum_{\substack{\eta \subset \omega, \eta \neq \varnothing \\ \eta' \subset \omega', \eta' \neq \varnothing}} V_{|\eta \vee \eta'|}(\eta \vee \eta') \tag{5.3}$$

and $Z_\Lambda^{\Gamma_\Lambda(\omega) \vee \omega'(\Lambda^c)}(z, \beta)$ is the normalization factor. For $(x)_n \subset \Omega(\partial_d(\Sigma_0))$, the corresponding correlation functions are defined to be eual to one if $(x)_n = \omega(\partial_d(\Sigma_0))$ and equal to zero otherwise.

Following the method of the ref. 37 (which I do not reproduce here), I derive the following identities that hold between the correlation functions (5.2):

$$\begin{aligned} &\rho_\Lambda^{\Gamma_\Lambda(\omega) \vee \omega'(\Lambda^c)}(z, \beta \mid (x_1)) \\ &\quad = z\chi_{\Lambda - \partial_{d^*}(\Sigma_0)}(x)_1 \exp -\beta\mathscr{E}((x)_1 \mid \Gamma_\Lambda(\omega) \vee \omega'(\Lambda^c)) \\ &\quad \times \sum_{k=0}^{\infty} \frac{1}{k!} \int_{\Gamma_\Lambda} dy_k \, \mathscr{K}(x_1 \mid \phi \mid (y)_k) \, \rho_\Lambda^{\Gamma_\Lambda(\omega) \vee \omega'(\Lambda^c)}(z, \beta \mid (y)_k) \end{aligned} \tag{5.4}$$

and for $n > 1$

$$\begin{aligned}
&\rho_\Lambda^{\Gamma_\Lambda(\omega) \vee \omega'(\Lambda^c)}(z, \beta \mid (x)_n) \\
&\quad = z\chi_{\Gamma_\Lambda}(x)_n \exp -\beta\mathscr{E}^1((x)_n \mid \Gamma_\Lambda(\omega) \vee \omega'(\Lambda^c)) \\
&\quad\quad \times \sum_{k=0}^{\infty} \frac{1}{k!} \int_{\Gamma_\Lambda} d(y)_k \, \mathscr{H}(x_1 \mid (x)'_n \mid (y)_k) \, \rho_\Lambda^{\Gamma_\Lambda(\omega) \vee \omega'(\Lambda^c)}(z, \beta \mid (y)_k \vee (x)'_n)
\end{aligned}$$

where the abbrevations as above have been used again. The kernel $\mathscr{H}$ is defined by

$$\mathscr{H}(x_1 \mid (x)_k \mid (y)_n) = \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} \sigma(x_1 \mid (x)_k \vee (y)_i) \tag{5.5}$$

$$\sigma(x \mid (x)_n) = \exp -\beta \sum_{\substack{(x)_q \subset (x)_n \\ q \geqslant 1}} \mathscr{E}((x) \vee (x)_q) \tag{5.6}$$

and finally

$$\mathscr{E}^1((x)_n \mid \omega) = \sum_{\substack{(x)_q \subset (x)_n: \\ (x)_1 \subset (x)_q}} \mathscr{E}((x)_q \mid \omega) \tag{5.7}$$

The generic operator $\mathbb{K}_\Lambda^{\Gamma_\Lambda(\omega) \vee \omega'(\partial(\Lambda^c))}$ for Eq. (5.4) acts in the space $\Pi_{\Gamma_\Lambda}\mathscr{B}_1$ according to

$$\begin{aligned}
&(\mathbb{K}_\Lambda^{\Gamma_\Lambda(\omega) \vee \omega'(\Lambda^c)} f)(x_1) \\
&\quad = z\Pi_\Lambda \Pi_{R^d - \partial_{d^*(\Sigma_0)}} \exp -\beta\mathscr{E}((x)_1 \mid \Gamma_\Lambda(\omega) \vee \omega'(\Lambda^c)) \\
&\quad\quad \times \sum_{k=1}^{\infty} \frac{1}{k!} \int d(y)_k \, \mathscr{H}(x_1 \mid \phi \mid (y)_k) \, \Pi_\Lambda(\Pi_{R^d - \partial_{d^*(\Sigma_0)}})(f)_k(y)_k \qquad (5.8) \\
&(K_\Lambda^{\Gamma_\Lambda(\omega) \vee \omega'(\Lambda^c)} f)_n(x)_n \\
&\quad = z\Pi_\Lambda \Pi_{R^d - \partial_{d^*(\Sigma_0)}} \exp -\beta\mathscr{E}^1((x)_n \mid \Gamma_\Lambda(\omega) \vee \omega'(\Lambda^c)) \\
&\quad\quad \times \sum_{k=1}^{\infty} \frac{1}{k!} \int d(y)_k \, \mathscr{H}(x_1 \mid (x)'_n \mid (y)_k)(\Pi_\Lambda \Pi_{R^d - \partial_{d^*(\Sigma_0)}} f_k)(y)_k
\end{aligned}$$

Defining

$$\begin{aligned}
&\alpha_\Lambda(\Gamma_\Lambda(\omega) \vee \omega'(\Lambda^c)) \\
&\quad = (\chi_{\Lambda - \partial_{d^*(\Sigma_0)}} \exp -\beta\mathscr{E}((x)_1 \mid \Gamma_\Lambda(\omega) \vee \omega'(\Lambda^c)), 0, \ldots, 0, \ldots) \qquad (5.9)
\end{aligned}$$

we can rewrite identities (5.3) as

$$\rho_\Lambda^{\Gamma_\Lambda(\omega)\vee\omega'(\Lambda^c)}(z,\beta) = z\mathbb{K}_\Lambda^{\Gamma_\Lambda(\omega)\vee\omega'(\Lambda^c)}\rho_\Lambda^{\Gamma_\Lambda(\omega)\vee\omega'(\Lambda c)}(z,\beta)+z\alpha_\Lambda(\Gamma_\Lambda(\omega)\vee\omega'(\Lambda^c)) \quad (5.10)$$

Lemma 5.1. Let $|z|<\exp[-2V_d(d^*)]$. Then for any $\omega\in\Omega^*$ the operator $K_{\Lambda=R^d}^{\Gamma_\infty(\omega)\vee\varnothing}$ is contractive in the space $\Pi_{R^d-\partial_{d^*}(\Sigma_0)}\mathscr{B}_1$ and Eq. (5.4) (with $\omega'=\varnothing$, $\Lambda=R^d$) has a unique solution $\rho_\infty^{\Gamma_\infty(\omega)}(z,\beta)\in\mathscr{B}_1$. Moreover, the conditioned correlation functions $\{\rho_\Lambda^{\Gamma_\infty(\omega)\vee\phi}(z,\beta\,|\,(x)_n\}_{n=1,2,\dots}$ tend to $\rho_\infty^{\Gamma_\infty(\omega)}(z,\beta)$ locally uniformly componentwise.

The proof is standard and will be omitted here.

Proposition 5.2. Let $|z|<\exp[-2V_d(d^*)]$. Then for every $\omega,\omega'\in\Omega$ we have the convergence

$$\rho_\infty^{\Gamma_\infty(\omega)}(z,\beta)=\lim_{\Lambda\uparrow R^d}\rho_\Lambda^{\Gamma_\Lambda(\omega)\vee\omega'(\Lambda^c)}(z,\beta) \quad (5.11)$$

where the limit is taken in the $*$-w topology of the space $\mathscr{B}_1$ and the convergence $\Lambda\uparrow R^d$ means the convergence in the van Hove sense.

Proof. The main arguments are the same as in the two-body potential case. We look for the dual to the operator $\mathbb{K}_\Lambda^{\Gamma_\Lambda(\omega)\vee\omega'(\Lambda^c))}$ in the dual Banach pair $(*\mathscr{B}_1,\mathscr{B}_1)$ and show the strong convergence of this dual to the dual of $K_\infty^{\Gamma_\infty(\omega)}$.

Modulo the multiplicative part, the dual of the operator $K_\Lambda^{\Gamma_\Lambda(\omega)\vee\omega'(\Lambda^c)}$ is given by

$$(*k_\Lambda\psi)(x)_n=\sum_{q=0}^{n}\binom{n}{q}(-1)^{n-q}\int_\Lambda dy\,\sigma(y\,|(x)_q|\,(x)_n-(x)_q) \times\psi_{1+n-q}(y\vee((x)_n-(x)_q)) \quad (5.12)$$

From this it is easy to conclude

$$s\text{-}\lim_{\Lambda\uparrow R^d}*k_\Lambda=*k_\infty \qquad (\text{in } \Pi_{R^d-\Gamma_\infty}\mathscr{B}_1) \quad (5.13)$$

For any compact $\Delta\subset R^d-\Gamma_\infty$ we have

$$\lim_{\Lambda\uparrow R^d}\|\mathscr{E}^1((x)_n\,|\,\Gamma_\Lambda(\omega)\vee\omega'(\Lambda^c))-\mathscr{E}^1((x)_n\,|\,\Gamma_\infty(\omega)\|_{L^\infty(\Delta)}=0 \quad (5.14)$$

From (5.13) and (5.14) it follows that

$$s\text{-}\lim_{\Delta\uparrow R^d}*K_\Lambda^{\Gamma_\Lambda(\omega)\vee\omega'(\Lambda^c))}=*\mathbb{K}_\infty^{\Gamma_\infty(\omega)}$$

in the space $*(\Pi_{R^d-\Gamma_\infty}\mathscr{B}_1)$.

The rest of the argument is the same as in Section 3. QED

Now by the method of the Appendix, we increase the proved *-w convergence (5.11) to the locally uniformly, componentwise sense, which yields weak convergence of the corresponding Gibbs measures and proves Theorem 1.

APPENDIX. FROM THE *-WEAK TO LOCALLY UINIFORM, COMPONENTWISE CONVERGENCE

Iterating the procedure which leads to the identities of the type (5.9), we get

$$\rho_{\Lambda}^{\Gamma_\Lambda(\omega) \vee \omega'(\Lambda^c)}(z, \beta) = \mathcal{M}_{\Lambda}^{\Gamma_\Lambda(\omega) \vee \omega'(\Lambda^c)}(z, \beta)\, \rho_{\Lambda}^{\Gamma_\Lambda(\omega) \vee \omega'(\Lambda^c)}(z, \beta) \tag{A.1}$$

where the operators $\mathcal{M}_{\Lambda}^{\Gamma_\Lambda(\omega) \vee \omega'(\Lambda^c)}$ are defined by

$$\begin{aligned}(\mathcal{M}_{\Lambda}^{\Gamma_\Lambda(\omega) \vee \omega'(\Lambda^c)} f)_n(x)_n &\\ = z^n \chi_{\Lambda - \partial_d(\Sigma_0)}(x)_n \exp &-\beta\mathscr{E}((x)_n \mid \omega'(\Lambda^c) \vee \Gamma_\Lambda(\omega))\\ \times \sum_{m=0}^{\infty} 1/m! &\int_{\Lambda - \partial_d(\Sigma_0)} d(y)_m\, M((x)_n \mid (y)_m)\, f_m((y)_m)\end{aligned} \tag{A.2}$$

where the kernels $M((x)_n \mid (y)_m)$ are given by

$$M((x)_p \mid (y)_q) = \sum_{l=0}^{q} \binom{q}{l} (-1)^{q-l} m((x)_p \mid (y)_l) \tag{A.3}$$

with

$$m((x)_p \mid (y)_m) = \exp -\beta \sum_{\substack{(x)_k \subset (x)_p \\ (y)_l \subset (y)_m \\ k \geq 1, l \geq 1}} V_{k+l}((x)_k \vee (y)_l) \tag{A.4}$$

Fixing the configuration $(x)_p$, we have that the maps

$$(x)_p \to M((x)_p \mid (-)_q)$$

are the maps from $(R^d - \partial_d(\Sigma_0))^{\otimes p}$ to the space $L^1((R^d - \partial_d(\Sigma_0))^{\otimes q})$ with the bound on the norm

$$\| M((x)_p \mid (-)_q) \|_{L^1(R^d - \partial_d(\Sigma_0))^{\otimes q}} \leq C p^q \tag{A.5}$$

where C is some constant. This bound comes from simple geometric considerations taking into account the assumed short-distance behavior of (V_k) and formulas (A.3) and (A.4). From (A.5) it then follows that the vector $\{1/q!\, M((x)_p\,|\,(x)_q)\}_{q=1,\dots}$ for each fixed $(x)_p$ belongs to the space $^*\beta_1$ and we have the estimate on its norm

$$^*\left\| \left\{ \frac{1}{q!} M((x)_p\,|\,(y)_q) \right\}_{q=1,\dots} \right\|_1 \leqslant \exp O(1)\, p \tag{A.6}$$

Iterating now Eq. (5.11) with $\Lambda = R^d$ and $\omega' = \varnothing$, we get similar identities for $\rho_\infty^{\omega(\partial_d(\Sigma_0))}(z, \beta)$.

Thus, we have

$$\begin{aligned}
&\rho_\Lambda^{\Gamma_\Lambda(\omega) \vee \omega'(\Lambda))}(z, \beta\,|\,(x)_n) - \rho_\infty^{\Gamma_\infty(\omega)}(z, \beta\,|\,(x)_n) \\
&\quad = z^n\{\exp -\beta\mathscr{E}((x)_n\,|\,\Gamma_\Lambda(\omega) \vee \omega(\Lambda^c)) \\
&\qquad - \exp -\beta\mathscr{E}((x)_n\,|\,\Gamma_\infty(\omega))\, \chi_{R^d - \partial_d(\Sigma_0)}(x)_n\} \\
&\qquad \times \sum_{m=1}^\infty \frac{1}{m!} \int_{(R^d - \partial_d(\Sigma_0))^{\otimes m}} d(y)_m\, M((x)_n\,|\,(y)_m)\, \rho_\infty^{\Gamma_\infty(\omega)}(z, \beta\,|\,(y)_m) \\
&\qquad + z^n \exp -\beta\mathscr{E}((x)_n\,|\,\Gamma_\Lambda(\omega) \vee \omega'(\Lambda^c))\, \chi_{R^d - \partial_d(\Sigma_0)}((x)_n) \\
&\qquad \times \sum_{m=1}^\infty \frac{1}{m!} \int_{(R^d - \partial_d(\Sigma_0))} d(y)_m\, M((x)_n\,|\,(y)_m) \\
&\qquad \times (\rho_\Lambda^{\Gamma_\Lambda(\omega) \vee \omega'(\Lambda^c)}(z, \beta\,|\,(y_m) - \rho_\infty^{\Gamma_\infty(\omega)}(z, \beta))
\end{aligned}$$

Now the claim follows easily from the proven *-weak convergence, fact (A.6), and the assumed decay of $(V_k)_k$. QED

ACKNOWLEDGMENTS

I am very grateful to Profs. S. Albeverio, P. Blanchard, and L. Streit for their invitation to join the BiBoS Research Project at the University of Bielefeld where this work was done.

REFERENCES

1. R. J. Baxter, *Exactly Solvable Models in Statistical Mechanics* (Academic Press, London, 1982).
2. M. O'Caroll and M. S. Schor, *Commun. Math. Phys.* **103**:1–33 (1986), and references therein.
3. J. Glimm and A. Jaffe, *Quantum Physics Functional Integral Point of View.*
4. Yu. M. Suhov, *Works Moscow Math. Soc.* **24** (1971) (in Russian)
5. W. J. Skripnik, *Doklady Akad. Nauk SSSR* **222**:795 (1975).
6. W. J. Skripnik, Preprint J.T.P. Kiev (1972).
7. R. Gielerak, *Comm. Ann. Inst. H. Poincare* **48**:205 (1988).
8. R. Gielerak, *J. Math. Phys.* **30**:115 (1989).

9. R. L. Minlos, *Funct. Anal. Appl.* **1** (1968).
10. S. Albeverio, R. Hoegh-Kröhn, and G. Olesen, *J. Multiv. Anal.* (1981); also preprint Bielefeld-Marseille (November 1978).
11. J. Bellisard and R. Hoegh-Kröhn, *Commun. Math. Phys.* **84**:297–327 (1982).
12. S. Goldstein, *Commun. Math. Phys.* **74**:223–234 (1980).
13. B. Zegarlinski, *J. Stat. Phys.* **43**:687–705 (1986).
14. S. Albeverio and R. Hoegh-Kröhn, *Commun. Math. Phys.* **68**:95 (1979).
15. R. Gielerak, *J. Math. Phys.* **24**:247 (1983); *Math. Phys.* **27**:1192 (1986).
16. R. Gielerak, in *Critical Phenomena. Theoretical Aspects* (Birkhauser, Boston, 1985).
17. R. Gielerak and B. Zegarlinski, *Fortschr. Phys.* **1**:1–24 (1984).
18. R. L. Dobrushin, *Theor. Prob. Appl.* **2**:197–224 (1968).
19. R. L. Dobrushin, *Funct. Anal. Appl.* **3**:27–35 (1969).
20. R. Gielerak, Transfer matrix for stable interactions, work in progress.
21. C. Preston, *Random Fields* (Springer-Verlag, New York, 1975).
22. H. O. Georgii, *Canonical Gibbs Measures* (Springer-Verlag, New York, 1979).
23. X. X. Ngyen and H. Zessin, *Math. Nachr.* **88**:105 (1979).
24. R. L. Dobrushin and E. A. Pecherski, in *Lecture Notes in Mathematics*, Vol. 1021 (Springer-Verlag, New York, 1983), p. 97.
25. R. L. Dobrushin, *Theor. Math. Phys.* **4**(1):101–118 (1970).
26. W. Klein, *Commun. Math. Phys.* **86**:227–246 (1982).
27. D. Ruelle, *Commun. Math. Phys.* **18**:127–159 (1970).
28. R. Gielerak, Comm. JINR E4-85-969 (1985); and in preparation.
29. H. von Weizräcker, in *Eighth Winter School on Abstract Analysis* (1980).
30. R. B. Israel, *Commun. Math. Phys.* **105**:669–673 (1986).
31. C. Kessler, *Publ. RIMS Kyoto* **24**:877–888 (1985).
32. H. Föllmer, in *Quantum Fields—Algebras, Processes*, L. Streit, ed. (Springer-Verlag, 1980).
33. Yu. M. Suhov, *Commun. Math. Phys.* **50**:113–132 (1976).
34. D. Ruelle, *Statistical Mechanics Rigorous Results* (Benjamin, New York, 1969).
35. N. Dunford and J. T. Schwartz, *Linear Operators*, Vol. I, *General Theory.*
36. V. A. Zagrebnov, *Theor. Math. Phys.* **51**(3):389–402 (1982).
37. H. Moraal, *Physica* **105**(A):286–296 (1981); *Phys. Lett.* **59A**(1):9 (1976).